Definition: (absolute value)

Let $A \in M_n(\mathbb{C})$.

Then A^*A is self-adjoint and positive semi-definite. By results in class and from HW #5, A^*A is unitarily diagonalizable,

$$\boxed{A^*A = WDW^*}$$

where W is unitary

and D is diagonal

with
$$D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$$

and $\boxed{d_i \geq 0}$ $\forall\ 1 \leq i \leq n$

Define the absolute value of A to be

$$\boxed{|A| = W \begin{pmatrix} \sqrt{d_1} & & & 0 \\ & \sqrt{d_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{d_n} \end{pmatrix} W^*}$$

Observation: $|A|$ is positive semi-definite and self-adjoint.

In fact, positive semi-definite implies self-adjoint!

Lemma: Let $A \in M_n(\mathbb{C})$

Then $\ker(|A|) = \ker(A)$.

proof: $x \in \ker(A)$

if and only if $Ax = 0$

if and only if $\|Ax\|_2^2 = 0$.

But $\|Ax\|_2^2$

$$= \langle Ax, Ax \rangle$$

$$= \langle x, A^*Ax \rangle$$

(properties of the adjoint)

$$= \langle x, |A|^2 x \rangle$$

(since $A^*A = |A|^2$)

$$= \langle |A|^* x, |A| x \rangle$$

$$= \langle |A| x, |A| x \rangle$$

(since $|A|$ self-adjoint)

$$= \| \, |A| x \|_2^2$$

So $\|Ax\|_2^2 = \| \, |A| x \|_2^2$.

Then

$$0 = \|Ax\|_2^2$$

$$= \| \, |A| x \|_2^2$$

if and only if

$$|A| x = 0$$

if and only if

$$x \in \ker(|A|).$$

This shows

$$\ker(|A|) = \ker(A). \quad \square$$

Observation: (invertibility)

Suppose A is invertible.

Then $0 = \ker(A)$

$= \ker(|A|)$

by previous lemma.

So $|A|$ is also

invertible.

We may define a unitary U by

$$\boxed{U = |A|^{-1} \cdot A^*.}$$

Note

$$UU^* = |A|^{-1}A^*\left(|A|^{-1}A^*\right)^*$$

$$= |A|^{-1}A^*A\left(|A|^{-1}\right)^*$$

$$= |A|^{-1}|A|^2|A|^{-1}$$

($|A|$ self-adjoint $\Rightarrow |A|^{-1}$ self-adjoint)

Then

$$UU^* = |A|^{-1} |A| \cdot |A| \cdot |A|^{-1}$$

$$= I_n$$

$\Rightarrow U$ is unitary.

Then

$$|A|U = |A| |A|^{-1} A^*$$

$$= A^*.$$

Finally,

$$A = (A^*)^*$$

$$= (|A| U)^*$$

$$= U^* |A|,$$

So we have a polar decomposition for A

provided A is invertible!

Theorem: (polar decomposition)

Let $A \in M_n(\mathbb{C})$. Then

$\exists$ unitary $U \in M_n(\mathbb{C})$

with

$$\boxed{A = U|A|}$$

Proof: Let $W \subseteq \mathbb{C}^n$,

$W = \operatorname{ran}(A)$.

Then we may decompose

$$\mathbb{C}^n = \operatorname{ran}(A) + (\operatorname{ran}(A))^{\perp}$$

(i.e. if $P: \mathbb{C}^n \to W$ is orthogonal projection, then $x \in \mathbb{C}^n$ may be written as $x = Px + P^{\perp}x$ with $P^{\perp}x \in W^{\perp}$ unique)

Define

$$\mathcal{V} : \operatorname{ran}(|A|) \to \operatorname{ran}(A)$$

by

$$\boxed{\mathcal{V}(|A|x) = Ax}$$

$$\forall \ x \in \mathbb{C}^n.$$

Check τ well-defined:

Suppose $|A|x = |A|y$

for some $x, y \in \mathbb{C}^n$.

$|A|$ is $\boxed{\text{linear}}$, so

$$|A|(x-y) = |A|x - |A|y$$
$$= 0$$

$$\Rightarrow x - y \in \ker(|A|)$$

Then $x - y \in \ker(|A|)$

$$= \ker(A)$$

by our lemma.

This implies

$$Ax - Ay = A(x-y) = 0.$$

Therefore

$$v(|A|x) = Ax = Ay = v(|A|y)$$

and so v is well-defined.

$\underline{\nu \text{ linear}}$ Let $x, y \in \mathbb{C}^n$.

$$\nu(|A|x + |A|y)$$

$$= \nu(|A|(x+y)) \quad (|A| \text{ linear})$$

$$= A(x+y)$$

$$= Ax + Ay \quad (A \text{ linear})$$

$$= \nu(|A|x) + \nu(|A|y) \quad \checkmark$$

Now let $\alpha \in \mathbb{C}$. Then

$$
\begin{aligned}
&v(\alpha |A| x) \\
&= v(|A|(\alpha x)) \quad (|A| \text{ linear}) \\
&= A(\alpha x) \\
&= \alpha A x \quad (A \text{ linear}) \\
&= \alpha v(|A| x) \quad \checkmark
\end{aligned}
$$

So v is linear.

V inner-product preserving

Let $x, y \in \mathbb{C}^n$.

$$\langle V(|A|x), V(|A|y)\rangle$$

$$= \langle Ax, Ay\rangle$$

$$= \langle x, A^*Ay\rangle$$

$$= \langle x, |A|^2 y\rangle$$

$$= \langle |A|x, |A|y\rangle$$

In particular, ν is injective since

$$0 = \| \nu(|A|x) \|_2^2$$

$$\iff \| |A| x \|_2^2 = 0$$

$$\iff |A| x = 0.$$

Moreover, ν is surjective by definition.

$$V : \mathrm{ran}(|A|) \to \mathrm{ran}(A)$$

is then an isomorphism of vector spaces

$$\Rightarrow \dim(\mathrm{ran}(|A|)) = \dim(\mathrm{ran}(A))$$

$$\Leftrightarrow \dim(\mathrm{ran}(|A|)^\perp) = \dim(\mathrm{ran}(A)^\perp)$$

Simply by counting dimensions since $\mathbb{C}^n$ is finite-dimensional!

Let $k = \dim(\operatorname{ran}(A)^{\perp})$

and let

$\{x_1, \ldots, x_k\}$,

$\{y_1, \ldots, y_k\}$ be

orthonormal bases of

$\operatorname{ran}(|A|)^{\perp}$ and $\operatorname{ran}(A)^{\perp}$,

respectively.

Define

$$S : \operatorname{ran}(|A|)^{\perp} \to \operatorname{ran}(A)^{\perp}$$

by
$$\boxed{S\left(\sum_{i=1}^{k} \alpha_i x_i\right) = \sum_{i=1}^{k} \alpha_i y_i}$$

$$\forall (\alpha_i)_{i=1}^{k} \subseteq \mathbb{C}.$$

Note: S linear and inner-product preserving.

Now let $x \in \mathbb{C}^n$ and write $x = Px + P^\perp x$ where $P : \mathbb{C}^n \to \operatorname{ran}(|A|)$ is orthogonal projection.

Define $U : \mathbb{C}^n \to \mathbb{C}^n$,

$$\boxed{U(x) = v(Px) + s(P^\perp x)}$$

U is linear by definition and well-defined since Px and $P^{\perp}x$ are unique.

Moreover,

$$\langle Ux, Uy \rangle$$

$$= \langle v(P_x) + S(P^{\perp}x), v(P_y) + S(P^{\perp}y) \rangle$$

$$= \langle v(P_x), v(P_y) \rangle$$

$$+ \langle v(P_x), S(P^{\perp}y) \rangle = 0$$

$$+ \langle S(P^{\perp}x), v(P_y) \rangle = 0$$

$$+ \langle S(P^{\perp}x), S(P^{\perp}y) \rangle$$

$$= \langle P_x, P_y \rangle + \langle P^{\perp}x, P^{\perp}y \rangle$$

$$= \langle x, y \rangle$$

This shows U is unitary! Finally,

$$U(|A|x) = Ax \quad \forall\, x \in \mathbb{C}^n$$

$$\Rightarrow U|A| = A.$$

$\square$

Observations:

1) "diagonalization"

Since $|A| = WDW^*$,

we can write $A \in \mathbb{C}^n$ as

$$A = U|A| = UWDW^*$$

$$= U_1 D W^*$$

untiaries

2) Uniqueness:

if $A = WP$ where W is unitary and P is positive semi-definite, then $\exists$ unitary V with $P = V|A|$ and $W = UV^*$.

3) Similarity to complex numbers

$$z \in \mathbb{C}, \quad z = e^{i\theta} |z|$$

where θ is the angle between z and the positive real axis.

$$U \sim e^{i\theta}$$

$$|A| \sim |z|$$

That's All!